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SINGULAR APPROXIMATION IN THE THEORY OF ELASTOPLASTIC MEDIA WITH MICROSTRUCTURE*

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Hypothesis of singular approximation is used in statistical averaging of the equations of equilibrium of a composite material with elastoplastic components to establish the upper estimation of the macroscopic defining relations. Formulas are obtained for determining the effective moduli of elasticity, the yield point and the linear kinematic strain hardening modulus.

1. Let us consider a two-component system, the components obeying the Hooke's law

$$\sigma_{ij}^{(1)} = 2\mu_1 (e_{ij}^{(1)} - e_{ij}^p) + \delta_{ij}\lambda_1 e_{kk}^{(1)} \tag{1.1}$$
$$\sigma_{ij}^{(2)} = 2\mu_2 e_{ij}^{(2)} + \delta_{ij}\lambda_2 e_{kk}^{(2)}$$

Here $\sigma_{ij}, e_{ij}, e_{ij}^p$ are the components of stress, total deformation and plastic deformation tensor, $\mu_\alpha, \lambda_\alpha$ $(\alpha = 1, 2)$ are the Lame' parameters of the components, and the plastic deformations satisfy the condition of incompressibility $e_{kk}^p = 0$. The plastic deformation of the material of the first phase is described in terms of the Mises yield surface

$$s_{ij}s_{ij} = k^2, \quad s_{ij} = \sigma_{ij} - {}^1/_3\delta_{ij}\sigma_{kk}$$

where k is the yield point. The structure of the composite whose components are joined under the condition of perfect adhesion, is characterised by the random, statistically homogeneous and isotropic indicator function of the coordinates $\varkappa(\mathbf{r})$ equal to zero at the points of the first phase, and to unity at the points of the second phase. The function, the stresses and the total and plastic deformations, are regarded as ergotically random fields, and their statistical means over the "ensemble" are replaced by the averages over the characteristic volumes /1/

$$\langle(\cdot)\rangle = \frac{1}{V}\int_V (\cdot)\,d\mathbf{r}, \quad \langle(\cdot)\rangle_\alpha = \frac{1}{V_\alpha}\int_{V_\alpha} (\cdot)\,d\mathbf{r}$$

where V is the total volume of the composite and V_α are the component volumes. The mechanical behavior of the material of the second component is assumed to be perfectly elastic, therefore /2/

$$\langle e_{ij}^p\rangle_2 = 0, \quad \langle\varkappa' e_{ij}^{p\prime}\rangle = -c_2\langle e_{ij}^p\rangle = -(1 - c_1)\langle e_{ij}^p\rangle \tag{1.2}$$

Here $c_\alpha = V_\alpha V^{-1}$ are the volume contents of the components and the primes denote the fluctuations in the values of the quantities over the whole volume of the body. Formulation of the Hook's Law written with help of the indicator function

$$\sigma_{ij} = 2(\langle\mu\rangle + [\mu]\varkappa')(e_{ij} - e_{ij}^p) + \delta_{ij}(\langle\lambda\rangle + [\lambda]\varkappa')e_{kk}$$

and equations of equilibrium of the composite medium $\sigma_{ij,j} = 0$ yield, after algebraic transformations,

$$2\langle\mu\rangle e_{ij,j}' + \langle\lambda\rangle e_{kk,i}' - 2\mu_1 e_{ij,j}^{p\prime} - (a_{ij}\varkappa' - \langle a_{ij}\varkappa'\rangle)_{,j} = 0$$
$$a_{ij} = -2[\mu]e_{ij} - \delta_{ij}[\lambda]e_{kk}, \quad [\mu] = \mu_2 - \mu_1, \ [\lambda] = \lambda_2 - \lambda_1$$

Supplementing this by the Cauchy relations $2e_{ij} = u_{i,j} + u_{j,i}$, we obtain a closed system of equations which reduces, with the help of the Green's tensor, to the equivalent system of integral equations /2/

$$e_{ij}(\mathbf{r}) = \langle e_{ij}\rangle + \int_V G_{i(k,l)j}(\mathbf{r} - \mathbf{r}_1)(2\mu_1 e_{kl}^{p\prime}(\mathbf{r}_1) + a_{kl}(\mathbf{r}_1)\varkappa'(\mathbf{r}_1) - \langle a_{kl}\varkappa'\rangle)\,d\mathbf{r}_1 \tag{1.3}$$

where $G_{ik}(\mathbf{r})$ is the Green's tensor, $u_i(\mathbf{r})$ is the displacement vector, and round brackets denote the symmetrizing over the indices.

Let us establish the relations connecting the deformations averaged over V_2 and V, restricting ourselves to the singular approximation /1/. To do this we replace the second

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derivative $G_{ik}(\mathbf{r})$ in (1.3) by its singular term, and taking into account (1.2) we obtain

$$\langle e_{ij}\rangle_2 = \langle e_{ij}\rangle - m_1\alpha\,\langle e_{ij}^p\rangle + \frac{c_1}{2\langle\mu\rangle}(\alpha\langle a_{ij}\rangle - \delta_{ij}\beta\langle a_{kk}\rangle) \tag{1.4}$$

Substituting (1.4) into (1.1) averaged over the whole volume, we obtain the macroscopic Hook's Law for the composite material

$$\langle s_{ij}\rangle = 2\mu^*(\langle d_{ij}\rangle - e_{ij}^*);\quad \langle\sigma_{kk}\rangle = 3K^*\langle e_{kk}\rangle \tag{1.5}$$

$$\mu^* = \xi\langle\mu\rangle,\quad \xi = 1 - \frac{c_1c_2[m]^2\alpha}{1+[m](c_1-c_2)\alpha}$$

$$K^* = \langle K\rangle - \frac{3c_1c_2[K]^2(\alpha-3\beta)}{2\langle\mu\rangle + (c_1-c_2)[K](\alpha-3\beta)}$$

$$m_\alpha = \frac{\mu_\alpha}{\langle\mu\rangle},\quad [m] = m_2 - m_1,\quad \alpha = \frac{2}{15}\frac{4-5\langle\nu\rangle}{1-\langle\nu\rangle},\quad \beta = \frac{1}{15(1-\langle\nu\rangle)}$$

$$\nu_\alpha = \frac{\lambda_\alpha}{2(\mu_\alpha+\lambda_\alpha)},\quad K_\alpha = \frac{2}{3}\mu_\alpha + \lambda_\alpha,\quad [K] = K_2 - K_1,$$

$$d_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}$$

Here μ^*, K^* are the effective shear moduli and all-directional tension-compression moduli, and e_{ij}^* are the residual deformations of the composite material measured after removing the external loads from the surface of the body. The deformations are connected with the mean plastic deformations by the relations

$$\langle e_{ij}^p\rangle = \frac{\xi}{m_1}e_{ij}^* \tag{1.6}$$

The expressions for the calculated elasticity moduli in the effective Hooke's Law obtained here coincide with the known formulas of singular approximation in the theory of elastic composites /1/.

2. Let us now determine the macroscopic strain hardening parameters of the composite medium beyond the limit of elasticity. To do this we average the local yield surface $\langle s_{ij}s_{ij}\rangle_1 = k^2$ over the volume V_1 of the first component. Using the condition that the square of the mean of a quantity is always smaller than the mean value of its squares, we obtain the upper estimate for the yield surface of the material within the volume V_1

$$\langle s_{ij}\rangle_1\langle s_{ij}\rangle_1 \leqslant k^2 \tag{2.1}$$

Using the Hooke's Law for the first component to eliminate from (2.1) the components of the stress tensor deviator and applying the rule of mechanical mixing of phases, we obtain

$$4\mu_1^2(\langle d_{ij}\rangle - \langle e_{ij}^p\rangle - c_2\langle d_{ij}\rangle_2)(\langle d_{ij}\rangle - \langle e_{ij}\rangle - c_2\langle d_{ij}^p\rangle_2) \leqslant k^2 \tag{2.2}$$

Substituting into (2.2) the formulas (1.2), (1.5) and (1.6), we obtain the upper estimates for the macroscopic load surface of the whole composite material

$$(\langle s_{ij}\rangle - Ne_{ij}^*)(\langle s_{ij}\rangle - Ne_{ij}^*) \leqslant k^{*2}$$

and the associated law of flow corresponding to this surface

$$\langle s_{ij}\rangle = k^*\frac{\varepsilon_{ij}^*}{\sqrt{\varepsilon_{mn}^*\varepsilon_{mn}^*}} + Ne_{ij}^*,\quad \varepsilon_{ij}^* = \frac{de_{ij}^*}{dt};\quad k^* = \xi\frac{k}{m_1} \tag{2.3}$$

$$N = \frac{2\mu_1}{c_1}\left(\frac{\xi}{m_1}\left(1 - \frac{c_2m_1\alpha}{1+[m](c_1-c_2)\alpha}\right) - c_1\mu^*\right)$$

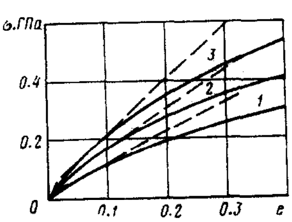

Fig.1

where k^* is the effective limit of plasticity of the composite and N is the linear, kinematic strain hardening coefficient. The relation (2.3) shows that under the conditions of perfectly plastic flow the whole composite behaves in volume V_1, on the whole, as a plastic body with linear kinematic strain hardening. The Fig.1 shows the comparison between the formulas (1.5) and (2.3), and the experimental /3/ uniaxial stretching of the samples of composite materials consisting of copper matrix base on sintered tungsten reinforcement. The solid lines correspond to the experiment and the dashed lines to the computations using the formulas (1.5) and (2.3). The curves *1, 2* and *3* correspond to the volume fractions of sintered tungsten reinforcement c_2, equal to 0.412; 0.512; 0.662.

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